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# Extending Drawings of Graphs to Arrangements of Pseudolines

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## Abstract

In the recent study of crossing numbers, drawings of graphs that can be extended to an arrangement of pseudolines (pseudolinear drawings) have played an important role as they are a natural combinatorial extension of rectilinear (or straight-line) drawings. A characterization of the pseudolinear drawings of  $K_n$  was found recently. We extend this characterization to all graphs, by describing the set of minimal forbidden subdrawings for pseudolinear drawings. Our characterization also leads to a polynomial-time algorithm to recognize pseudolinear drawings and construct the pseudolines when it is possible.

**2012 ACM Subject Classification** Mathematics of computing → Graph algorithms; Mathematics of computing → Graphs and surfaces

**Keywords and phrases** graphs, graph drawings, geometric graph drawings, arrangements of pseudolines, crossing numbers, stretchability.

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## 1 Introduction

Since 2004, geometric methods have been used to make impressive progress for determining the crossing number of (certain classes of drawings of) the complete graph  $K_n$ . In particular, drawings that extend to straight lines, or, more generally, arrangements of pseudolines, have been central to this work, spurring interest in such drawings for arbitrary graphs, not just complete graphs [2, 4, 5, 6, 12].

In particular, for pseudolinear drawings, it is now known that, for  $n \geq 10$ , a pseudolinear drawing of  $K_n$  has more than

$$H(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

crossings [1, 14]. The number  $H(n)$  is conjectured by Harary and Hill to be the smallest number of crossings over all topological drawings of  $K_n$ ; that is, the crossing number  $\text{cr}(K_n)$  is conjectured to be  $H(n)$ .

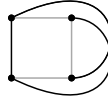
A *pseudoline* is the image  $\ell$  of a continuous injection from the real numbers  $\mathbb{R}$  to the plane  $\mathbb{R}^2$  such that  $\mathbb{R}^2 \setminus \ell$  is not connected. An *arrangement of pseudolines* is a set  $\Sigma$  of

pseudolines such that, if  $\ell, \ell'$  are distinct elements of  $\Sigma$ , then  $|\ell \cap \ell'| = 1$  and the intersection is a crossing point. Informally, a *crossing point* or *crossing* is an intersection point between two pseudolines that locally looks like a crossing point between two non parallel lines (a formal definition of crossing will be given when we introduce the notion of string). More on pseudolines and their importance for studying geometric drawings of graphs can be found in [10, 11].

A drawing  $D$  of a graph  $G$  is *pseudolinear* if there is an arrangement of pseudolines consisting of a different pseudoline  $\ell_e$  for each edge  $e$  of  $G$  and such that  $D[e] \subseteq \ell_e$ .

In the study of crossing numbers, restricting the drawing to either straight lines or pseudolines yields the rectilinear crossing number  $\overline{\text{cr}}(K_n)$  or the pseudolinear crossing number  $\widetilde{\text{cr}}(K_n)$ , respectively. Clearly  $\overline{\text{cr}}(K_n) \geq \widetilde{\text{cr}}(K_n)$  and the geometric methods prove that  $\widetilde{\text{cr}}(K_n) > H(n)$ , for  $n \geq 10$ .

A *good drawing* is one where no edge self-intersects and any two edges share at most one point—either a crossing or a common end point—and no three edges share a common crossing. One somewhat surprising result is from Aichholzer et al.: *a good drawing of  $K_n$  in the plane is homeomorphic to a pseudolinear drawing if and only if it does not contain a non-planar drawing of  $K_4$  whose crossing is incident with the unbounded face of the  $K_4$*  [2] (see Figure 1). By ignoring the grey edges from Figure 1, we see that any such drawing of  $K_4$  contains a *B-configuration*, depicted as the third drawing of the first row of Figure 2. Based on our Theorem 2, Theorem 2.5.1 from [3] shows that any non-pseudolinear drawing contains a *B-configuration*. Thus, either Fig. 1 or the *B-configuration* can be used to characterize pseudolinear drawings of  $K_n$ . In [4] pseudolinear drawings of  $K_n$  are characterized as *f-convex*, and in [5] are characterized as monotone and free of a specific drawing of  $K_4$ .



■ **Figure 1** Non-pseudolinear  $K_4$  with its crossing incident with the outer face.

Twenty-five years earlier, Thomassen [19] proved a similar theorem for a 1-planar drawing (that is, a drawing in which each edge is crossed at most once). The *B*- and *W*-configurations are shown as the third and fourth drawings in the first row of Figure 2. Thomassen's theorem is: *if  $D$  is a 1-planar drawing of graph  $G$ , then  $D$  is homeomorphic to a rectilinear drawing of  $G$  if and only if  $D$  contains no *B*- or *W*-configuration*.

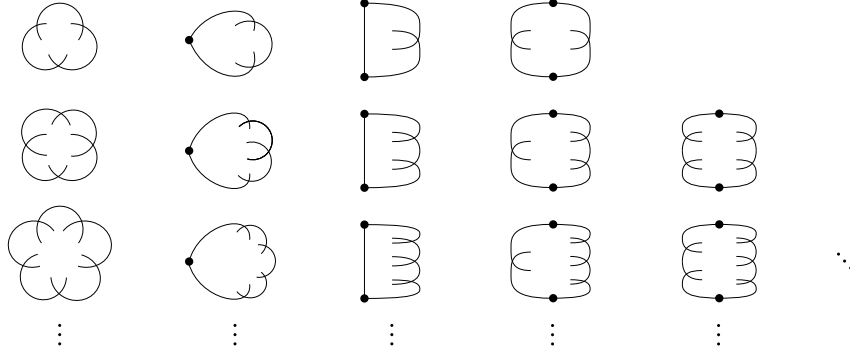
Thomassen presented in [19] the *clouds* (first column in Figure 2) as an infinite family of drawings that are minimally non-pseudolinear.

Shortly after Thomassen's paper, Bienstock and Dean proved that if  $\text{cr}(G) \leq 3$ , then  $\overline{\text{cr}}(G) = \text{cr}(G)$  [7]. They also exhibited examples based on overlapping *W*-configurations to show the result fails for  $\text{cr}(G) = 4$ ; such graphs can have arbitrarily large rectilinear crossing number.

Despite the existence of infinitely many obstructions to pseudolinearity, we characterize them all.

► **Theorem 1.** A good drawing of a graph  $G$  is pseudolinear if and only if it does not contain one of the infinitely many obstructions shown in Figure 1.

The drawings in Figure 2 are obtained from the *clouds* (first column) by replacing at most two crossings by vertices. The formal statement of Theorem 1 is Theorem 15 in Section 6; also a more general version of this statement, Theorem 2, is discussed below. Our result draws a



■ **Figure 2** Obstructions to pseudolinearity.

line between the class of pseudolinear drawings and the class of rectilinear drawings: Our result shows that recognizing pseudolinear drawings is a combinatorial/topological problem and implies a polynomial-time algorithm to detect pseudolinear drawings (Theorem 14). This contrast with the rather real algebraic geometry problem of deciding the *stretchability* of a drawing, defined as the problem of deciding whether a given drawing is homeomorphic to a rectilinear drawing. Mněv [16, 17] showed that deciding the stretchability of an arrangement of pseudolines is  $\exists\mathbb{R}$ -hard, implying the  $\exists\mathbb{R}$ -hardness for the problem of deciding the stretchability of a graph drawing. Since  $\text{NP} \subseteq \exists\mathbb{R}$  [15, 18, 8], this in particular shows that the stretchability problem is  $\text{NP}$ -hard. We refer to Matoušek’s survey [15] for an approachable introduction to the complexity class  $\exists\mathbb{R}$ .

The natural setting for our characterization is strings embedded in the plane. An *arc*  $\sigma$  is the image  $f([0, 1])$  of the compact interval  $[0, 1]$  under a continuous map  $f : [0, 1] \rightarrow \mathbb{R}^2$ . Let  $S(\sigma) = \{p \in \sigma : |f^{-1}(p)| \geq 2\}$  be the set of self-intersections of  $\sigma$ . A *string* is an arc  $\sigma$  for which  $S(\sigma)$  is finite. If  $S(\sigma) = \emptyset$ , then  $\sigma$  is *simple*. If  $\sigma'$  is a string and  $\sigma' \subseteq \sigma$ , then  $\sigma'$  is a substring of  $\sigma$ .

Suppose that  $\sigma$  and  $\sigma'$  whose intersection  $\sigma \cap \sigma'$  is a finite set and let  $p \in \sigma \cap \sigma'$ . The *rotation at  $p$*  is a cyclic sequence of substrings determined by a small neighbourhood homeomorphic to the plane in which  $p$  is origin and the substrings incident with  $p$  are rays emanating from  $p$  [13, Thm. 3.1]. The strings  $\sigma_1, \sigma_2$  cross at  $p$  if they each have two substrings that alternate  $\sigma_1 - \sigma_2 - \sigma_1 - \sigma_2$  in the rotation at  $p$ .

An intersection point between of two strings  $\sigma$  and  $\sigma'$  is *ordinary* if it is either an endpoint of  $\sigma$  or  $\sigma'$ , or is a *crossing*. A set  $\Sigma$  of strings is *ordinary* if  $\Sigma$  is finite and any two strings in  $\Sigma$  have only finitely many intersections, all of which are ordinary. All the sets of strings considered in this paper are ordinary.

If  $\Sigma$  is an ordinary set of strings, then its *planarization*  $G(\Sigma)$  is the plane graph obtained from  $\Sigma$  by inserting vertices at each crossing between strings and also at the endpoints of every string in  $\Sigma$ . To keep track of the information given by the strings, we will always assume that each string  $\sigma$  has a different color and that each edge in  $G(\Sigma)$  inherits the color of the string including it.

If  $\Sigma$  is an ordinary set of strings, then, for a cycle  $C$  in  $G(\Sigma)$  (which is a simple closed curve in  $\mathbb{R}^2$ ) the *edges inside  $C$*  are those drawn in the closed disk bounded by  $C$  (this includes the edges of  $C$ ). A vertex  $v \in V(C)$  is a *rainbow* for  $C$  if all the edges incident with  $v$  and drawn inside  $C$  have different colours. The reader can verify that, for each drawing in Figure 2, if we let  $\Sigma$  be the edges of the drawing, then the unique cycle in  $G(\Sigma)$  has at most two rainbows. Our main result characterizes these cycles as the only possible obstructions:

► **Theorem 2.** *An ordinary set of strings  $\Sigma$  can be extended to an arrangement of pseudolines if and only if every cycle  $C$  of  $G(\Sigma)$  has at least three rainbows.*

Henceforth, we define any cycle  $C$  in  $G(\Sigma)$  with at most two rainbows as an *obstruction*. A set of strings is *pseudolinear* if it has an extension to an arrangement of pseudolines.

Theorem 2 is our main contribution. In the next section, we show that the presence of an obstruction implies the set of ordinary strings is not pseudolinear. The converse is proved in Section 4 by extending, one small step at a time, the strings in  $\Sigma$  to get closer to an arrangement of pseudolines. After each extension, we must show that no obstruction has been introduced. This involves dealing with cycles in  $G(\Sigma)$  that have precisely three rainbows (that we refer as *near-obstructions*). In Section 3 we show the key lemma that if  $G$  has two such near-obstructions that intersect nicely at a vertex  $v$ , then  $G$  has an obstruction. In Section 5 we present a polynomial-time algorithm for detecting obstructions and we argue why the proof of Theorem 2 implies a polynomial-time algorithm for extending a pseudolinear set of strings. Finally, in Section 6, we show how Theorem 1 follows from Theorem 2 and we present some concluding remarks.

## 2 A set of strings with an obstruction is not extendible

Let us start by showing the easy direction of Theorem 2:

► **Lemma 3.** *If the underlying graph  $G(\Sigma)$  of a set  $\Sigma$  of strings has an obstruction, then  $\Sigma$  is not pseudolinear.*

Suppose that  $C$  is a cycle of  $G(\Sigma)$  for some set of strings  $\Sigma$ . We define  $\delta(C)$  as the set of vertices of  $C$  for which their two incident edges in  $C$  have different colours. In a set  $\Sigma$  of simple strings where no two intersect twice,  $|\delta(C)| \geq 3$  for every cycle  $C$  of  $G(\Sigma)$ .

► **Lemma 4.** *Let  $\Sigma$  be a set of simple strings where every pair intersect at most once. Suppose that  $C$  is an obstruction with  $|\delta(C)|$  as small as possible. Let  $S = x_0, x_1, \dots, x_\ell$  be a path of  $G(\Sigma)$  representing a substring of some string  $\sigma \in \Sigma$  such that  $x_0x_1 \in E(C)$ ,  $x_1 \in \delta(C)$  and  $x_1$  is not a rainbow of  $C$ . Then  $V(C) \cap V(S) = \{x_0, x_1\}$ .*

**Proof.** By way of contradiction, suppose that there is a vertex  $x_r \in V(C) \cap V(S)$  with  $r \geq 3$ . Assume that  $r \geq 3$  is as small as possible. Let  $P$  be the subpath of  $S$  connecting  $x_1$  to  $x_r$ . The facts  $x_0x_1 \in E(C)$ ,  $x_1 \in \delta(C)$ , and  $P \subseteq \sigma$  imply that  $x_1x_2 \notin E(C)$ . Because  $x_1$  is not a rainbow for  $C$  and no two strings tangentially intersect at  $x_1$ , the edge  $x_1x_2$  is drawn in the closed disk bounded by  $C$ . By choice of  $r$ ,  $P$  is an arc connecting  $x_1$  to  $x_r$  in the interior of  $C$ .

Let  $C_1$  and  $C_2$  be the two cycles of  $C \cup P$  containing  $P$ , labelled so that  $x_0x_1 \in E(C_1)$ . We shall use the minimality of  $|\delta(C)|$  to show that  $C_1$  and  $C_2$  are not obstructions. Then, we will count rainbows in  $C_1$  and  $C_2$  to obtain the contradiction that  $C$  is not an obstruction.

For a cycle  $X$ , let  $\rho(X)$  be the set of rainbows of  $X$ . For  $i = 1, 2$ , let  $Q_i = V(C_i) \setminus V(P)$ . As the edges of  $S$  are included in the same string, we see that  $\rho(C_1) \setminus Q_1 \subseteq \{x_r\}$  and  $\rho(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$ . Likewise,  $\delta(C_1) \setminus Q_1 \subseteq \{x_r\}$  and  $\delta(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$ .

Let us show that  $C_1$  and  $C_2$  are not obstructions. Because  $|\delta(C_2)| \geq 3$  and  $\delta(C_2) \setminus Q_2 \subseteq \{x_1, x_r\}$ ,  $|\delta(C) \cap Q_2| \geq 1$ . Since  $\delta(C_1) \setminus Q_1 \subseteq \{x_r\}$  and  $x_1 \in \delta(C)$ ,  $|\delta(C_1)| \leq |\delta(C_1) \cap Q_1| + |\{x_r\}| \leq |\delta(C)| - 2 + |\{x_r\}| < |\delta(C)|$ . Because  $|\delta(C_1)| \geq 3$  and  $|\delta(C_1) \setminus Q_1| \leq 1$ ,  $|\delta(C) \cap Q_1| \geq 2$ . Since  $x_1 \in \delta(C) \cap \delta(C_2)$ ,  $|\delta(C_2)| \leq |\delta(C) \cap Q_2| + |\{x_1, x_r\}| \leq |\delta(C)| - 3 + |\{x_1, x_r\}| < |\delta(C)|$ . Thus, neither  $C_1$  nor  $C_2$  is an obstruction.

159 Finally, as  $|\rho(C_1)| \geq 3$  and  $|\rho(C_1) \setminus Q_1| \leq 1$ ,  $|\rho(C) \cap Q_1| = |\rho(C_1) \cap Q_1| \geq 2$ . Because  
 160  $|\rho(C_2)| \geq 3$  and  $|\rho(C_2) \setminus Q_2| \leq 2$ ,  $|\rho(C) \cap Q_2| = |\rho(C_2) \cap Q_2| \geq 1$ . Thus  $|\rho(C)| \geq 3$ , a  
 161 contradiction.  $\blacktriangleleft$

162 **Proof of Lemma 3.** By way of contradiction, suppose that  $\Sigma$  is pseudolinear and that  $G(\Sigma)$   
 163 has an obstruction  $C$ .

164 Consider an extension of  $\Sigma$  to an arrangement of pseudolines, and then cut off the two  
 165 infinite ends of each pseudoline to obtain a set of strings  $\Sigma'$  extending  $\Sigma$ , and in which every  
 166 pair of strings in  $\Sigma'$  cross once. In  $G(\Sigma')$ , there is a cycle  $C'$  that represents the same simple  
 167 closed curve as  $C$ . Because every rainbow of  $C'$  is a rainbow of  $C$ ,  $C'$  has fewer than three  
 168 rainbows. Therefore, we may assume that  $\Sigma = \Sigma'$  and  $C = C'$ . Now, the ends of every string  
 169 in  $\Sigma$  are degree-1 vertices in the outer face of  $G(\Sigma)$ .

170 As every string in  $\Sigma$  is simple and no two strings intersect more than once,  $|\delta(C)| \geq 3$ .  
 171 We will assume that  $C$  is chosen to minimize  $|\delta(C)|$ .

172 Since  $C$  is an obstruction, there exists  $x_1 \in \delta(C)$  such that  $x_1$  is not a rainbow in  $C$ .  
 173 Consider a neighbour  $x_0$  of  $x_1$  in  $C$ . Let  $S = x_0, x_1, \dots, x_\ell$  be the path obtained by traversing  
 174 the string  $\sigma$  extending  $x_0x_1$ , such that  $x_\ell$  is an end of  $\sigma$ . By Lemma 4,  $V(S) \cap V(C) = \{x_0, x_1\}$ ,  
 175 and because  $x_\ell$  is in the outer face of  $C$ , the segment of  $\sigma$  from  $x_1$  to  $x_\ell$  has its relative  
 176 interior in the outer face of  $C$ .

177 However, since  $x_1$  is not a rainbow, there exists a string  $\sigma' \in \Sigma$  including two edges at  $x_1$   
 178 drawn inside  $C$ . Thus,  $\sigma$  and  $\sigma'$  tangentially intersect at  $x_1$ , a contradiction.  $\blacktriangleleft$

### 179 **3 The key lemma**

180 In this section we present the key lemma used in the proof of Theorem 2.

181 A plane graph  $G$  is *path-partitioned* if for  $m \geq 1$ , there exists a colouring  $\chi : E(G) \rightarrow$   
 182  $\{1, \dots, m\}$  such that for each  $i \in \{1, \dots, m\}$ , the edges in  $\chi^{-1}(i)$  induce a path  $P_i \subseteq G$  where  
 183 any two distinct paths  $P_i$  and  $P_j$  do not tangentially intersect. Indeed, every underlying  
 184 planar graph  $G(\Sigma)$  of a set of simple strings  $\Sigma$  is path-partitioned. Moreover, every path-  
 185 partitioned plane graph can be obtained by subdividing a planarization of an ordinary set of  
 186 simple strings. To extend the previously introduced notation we refer to each  $P_i$  as a string.  
 187 The concepts of rainbow and obstruction naturally extend to the context of path-partitioned  
 188 plane graphs.

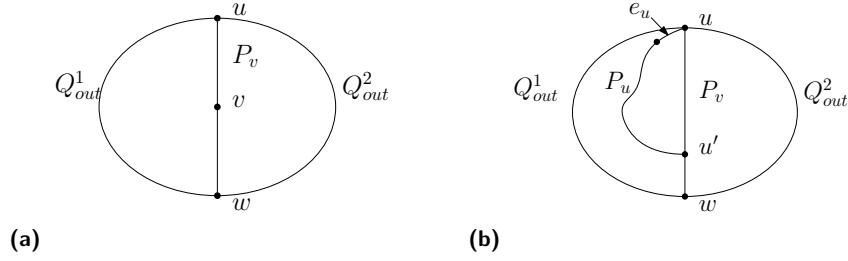
189 Suppose that  $G$  is a path-partitioned plane graph. Given  $v \in V(G)$ , a *near-obstruction at*  
 190  $v$  is a cycle  $C$  with at most three rainbows and such that  $v$  is a rainbow of  $C$ . Understanding  
 191 how near-obstructions behave is the key ingredient needed in the proof of Theorem 2:

192 **► Lemma 5.** *Let  $G$  be a path-partitioned plane graph and let  $v \in V(G)$ . Suppose that  $C_1$   
 193 and  $C_2$  are two near-obstructions at  $v$  such that the union of the closed disks bounded by  $C_1$   
 194 and  $C_2$  contains a small open ball centered at  $v$ . Suppose that one of the following two holds:*

- 195 1. *no obstruction of  $G$  contains  $v$ ; or*
- 196 2. *the two edges of  $C_1$  incident with  $v$  are the same as the two edges of  $C_2$  incident with  $v$ .*

197 *Then  $G$  has an obstruction not including  $v$ .*

198 Given a plane graph  $G$ , a cycle  $C \subseteq G$  and a vertex  $v \in V(C)$ , the edges at  $v$  inside  $C$  are  
 199 the edges of  $G$  incident with  $v$  drawn inside  $C$ . Consider a homeomorphism from a small  
 200 disc neighbourhood of  $v$  to the plane so that each edge segment incident with  $v$  is a straight  
 201 ray from the origin (which is  $v$ ). Since no two strings intersect tangentially at  $v$ , we may  
 202 assume that the rotation at  $v$  has substrings of the same colour making an angle of  $\pi$  at  $v$ .



■ **Figure 3** Auxiliary figures used in the proof of Lemma 5.

203 The angles between rays are the *angles at  $v$*  and we associate to them the set of edges at  $v$   
 204 drawn as rays inside them. From this geometric perspective, it is obvious that, if an angle  $\alpha$   
 205 is less than  $\pi$ , then  $\alpha$  is rainbow. This proves the second of the following facts.

206 ► **Useful Facts.** *Let  $G$  be a plane path-partitioned graph and let  $v \in V(G)$ . Then*

- 207 1. *if  $\alpha, \beta$  are two angles at  $v$  with  $\alpha \subseteq \beta$  and  $\beta$  is rainbow, then  $\alpha$  is rainbow; and*
- 208 2. *if  $\alpha$  and  $\beta$  are two angles such  $\bar{\alpha}$  is not rainbow and  $\beta$  is a proper subangle of the*  
 209 *complement  $\bar{\alpha}$  of  $\alpha$ , then  $\beta$  is rainbow.*

210 **Proof of Lemma 5.** By way of contradiction, suppose that  $G$  has no obstruction not includ-  
 211 ing  $v$ . The “small ball” hypothesis implies that  $v$  is not in the outer face of the subgraph  
 212  $C_1 \cup C_2$ .

213 We claim that  $|V(C_1) \cap V(C_2)| \geq 3$ . Suppose not. For  $i = 1, 2$ , let  $e_i$  and  $f_i$  be the edges  
 214 of  $C_i$  at  $v$  and let  $\Delta_i$  be the closed disk bounded by  $C_i$ . From the “small ball” hypothesis it  
 215 follows that (i)  $\Delta_1$  contains the edges  $e_2$  and  $f_2$ ; and (ii) the points near  $v$  in the exterior of  
 216  $\Delta_2$  are contained in  $\Delta_1$ . These two properties imply that the path  $C_2 - v$  intersects  $C_1$  at  
 217 least twice, and because  $v \in V(C_1) \cap V(C_2)$ ,  $|V(C_1) \cap V(C_2)| \geq 3$ .

218 From the last paragraph we know that  $C_1 \cup C_2$  is 2-connected, and hence the outer face  
 219 of  $C_1 \cup C_2$  is bounded by a cycle  $C_{out}$ . We will assume that

- 220 (\*) the cycles  $C_1$  and  $C_2$  satisfying the hypothesis of Lemma 5 are chosen so that the number  
 221 of vertices of  $G$  in the disk bounded by  $C_{out}$  is minimal.

222 Useful Fact 1 applied to the interior angles at vertices of  $C_{out}$  shows that every vertex  
 223 that is a rainbow in  $C_{out}$  is also a rainbow in each of the cycles in  $\{C_1, C_2\}$  containing it.  
 224 We can assume that  $C_{out}$  is not an obstruction or else we are done. We may relabel  $C_1$  and  
 225  $C_2$  so that two of the rainbows of  $C_{out}$ , say  $p$  and  $q$ , are also rainbows in  $C_1$ . Neither  $p$  nor  $q$   
 226 is  $v$  because  $v \notin V(C_{out})$ . Because  $C_1$  is a near-obstruction,  $p, q$  and  $v$  are the only rainbows  
 227 of  $C_1$ .

228 Since  $v \notin V(C_{out})$ , by following  $C_1$  in the two directions starting at  $v$ , we find a path  
 229  $P_v \subseteq C_1$  containing  $v$  in which only the ends  $u$  and  $w$  of  $P_v$  are in  $C_{out}$  (note that  $u \neq w$   
 230 because  $\{p, q\} \subseteq V(C_1) \cap V(C_{out})$ ). See Figure 3a.

231 As  $v$  is in the interior face of  $C_{out}$ ,  $P_v$  is also in the interior of  $C_{out}$ . Let  $Q_{out}^1, Q_{out}^2$  be  
 232 the  $uw$ -paths of  $C_{out}$ . One of the two closed disks bounded by  $P_v \cup Q_{out}^1$  and  $P_v \cup Q_{out}^2$   
 233 contains  $C_1$ . By symmetry, we may assume that  $C_1$  is contained in the first disk. Since  
 234  $C_{out} \subseteq C_1 \cup C_2$ , this implies that  $Q_{out}^2$  is a subpath of  $C_2$ .

235 Our desired contradiction will be to find three rainbows in  $C_2$  distinct from  $v$ . We  
 236 find the first: let  $C_1 - (P_v)$  be the  $uw$ -path in  $C_1$  distinct from  $P_v$ . The disk bounded by



(237)  $(C_1 - (P_v)) \cup Q_{out}^2$  contains the one bounded by  $C_1$ . Useful Fact 1 applied to the interior  
 (238) angles at the vertices of  $(C_1 - (P_v)) \cup Q_{out}^2$  implies that each vertex in  $C_1 - (P_v)$  that is a  
 (239) rainbow in  $(C_1 - (P_v)) \cup Q_{out}^2$  is also rainbow in  $C_1$ . Since  $C_1$  has at most two rainbows in  
 (240)  $C_1 - (P_v)$ , namely  $p$  and  $q$ ,  $(C_1 - (P_v)) \cup Q_{out}^2$  has a third rainbow  $r_1$  in the interior of  $Q_{out}^2$   
 (241) (else  $(C_1 - (P_v)) \cup Q_{out}^2$  is an obstruction and we are done). Note that  $r_1$  is also a rainbow  
 (242) for  $C_2$ .

(243) To find another rainbow in  $C_2$ , consider the edge  $e_u$  of  $C_2$  incident to  $u$  and not in  $Q_{out}^2$ .  
 (244) We claim that either  $u$  is a rainbow in  $C_2$  or that  $e_u$  is not included in the closed disk  
 (245) bounded by  $P_v \cup Q_{out}^2$ . Seeking a contradiction, suppose that  $u$  is not a rainbow of  $C_2$  and  
 (246) that  $e_u$  is included in the disk. Then Useful Fact 2 implies that  $u$  is a rainbow in  $C_1$ . As  $p$   
 (247) and  $q$  are the only rainbows of  $C_1$  in  $C_{out}$ ,  $u$  is one of  $p$  and  $q$ . Therefore  $u$  is a rainbow in  
 (248)  $C_{out}$ , and hence, a rainbow in  $C_2$ , a contradiction.

(249) If  $u$  is a rainbow in  $C_2$ , then this is the desired second one. Otherwise,  $e_u$  is not in the  
 (250) closed disk bounded by  $P_v \cup Q_{out}^2$ . Let  $P_u \subseteq C_2$  be the path starting at  $u$ , continuing on  $e_u$   
 (251) and ending on the first vertex  $u'$  in  $P_v$  that we encounter. Let  $C_u$  be the cycle consisting of  
 (252)  $P_u$  and the  $uu'$ -subpath  $uP_vu'$  of  $P_v$ . See Figure 3b.

(253)  $\triangleright$  **Claim 6.** If  $P_u$  does not have a rainbow of  $C_u$  in its interior, then either  $C_u$  is an  
 (254) obstruction not containing  $v$  or:

- (255) (a)  $C_u$  and  $C_2$  are near-obstructions at  $v$  satisfying the same conditions as  $C_1$  and  $C_2$  in  
 (256) Lemma 5; and
- (257) (b) the closed disk bounded by the outer cycle of  $C_u \cup C_2$  contains fewer vertices than the  
 (258) disk bounded by  $C_{out}$ .

(259) **Proof.** Suppose that all the rainbows of  $C_u$  are located in  $uP_vu'$ . If  $z$  is a rainbow of  $C_u$ ,  
 (260) then  $z \in \{u, v, u'\}$ , as otherwise  $z$  is a rainbow of  $C_1$  distinct from  $p, q$  and  $v$ , a contradiction.  
 (261) Thus, if  $v \notin V(C_u)$ , then  $C_u$  is the desired obstruction. We may assume that  $v \in V(C_u)$ .

(262) If  $u' = w$ , then  $C_2 = P_u \cup Q_{out}^2$ , violating the assumption that  $v \in V(C_2)$ . Thus  $u' \neq w$ .  
 (263) If  $u' = v$ , then the rainbows of  $C_u$  are included in  $\{u, u'\}$ , and hence  $C_u$  is an obstruction.  
 (264) However, the existence of  $C_u$  shows that both alternatives (1) and (2) in Lemma 5 fail:  
 (265) condition (1) fails because  $C_u$  contains  $v$  and (2) fails because the edge of  $P_u$  incident with  $v$   
 (266) is in  $E(C_2) \setminus E(C_1)$ . Thus  $u' \neq v$ .

(267) The previous two paragraphs show that  $C_u$  is a near-obstruction at  $v$  with rainbows  $u$ ,  
 (268)  $v$  and  $u'$ . Since the interior of  $C_u$  near  $v$  is the same as the interior of  $C_1$  near  $v$ , the pair  
 (269)  $(C_u, C_2)$  satisfies the “small ball” hypothesis. Thus, (a) holds.

(270) Let  $C'_{out}$  be the outer cycle of  $C_u \cup C_2$ . From the fact that  $C_u \cup C_2 \subseteq C_1 \cup C_2$  it follows  
 (271) that the disk bounded by  $C_{out}$  includes the disk bounded by  $C'_{out}$ .

(272) Since  $p, q \in V(C_{out})$ ,  $p$  and  $q$  are in the disk bounded by  $C_{out}$ . If both  $p$  and  $q$  are in  
 (273)  $C_2$ , then  $p, q$  and  $r_1$  are rainbows in  $C_2$ , and also distinct from  $v$ , contradicting that  $C_2$  is a  
 (274) near-obstruction for  $v$ . If, say  $p \notin V(C_2)$ , then  $p$  is not in the disk bounded by  $C'_{out}$ , which  
 (275) implies (b).  $\blacktriangleleft$

(276) From Claim 6(b) and assumption (\*) either  $C_u$  is the desired obstruction or  $P_u$  contains  
 (277) a rainbow  $r_2$  of  $C_2$  in its interior. We assume the latter as otherwise we are done.

(278) In the same way, the last rainbow  $r_3$  comes by considering the edge of  $C_2 - Q_{out}^2$  incident  
 (279) with  $w$ . It follows that  $v, r_1, r_2$  and  $r_3$  are four different rainbows in  $C_2$ , contradicting the  
 (280) fact that  $C_2$  is a near-obstruction.  $\blacktriangleleft$



## 4 Proof of Theorem 2

In this section we prove that a set of strings with no obstructions can be extended to an arrangement of pseudolines.

**Proof of Theorem 2.** It was shown in Observation 3 that the existence of obstructions implies non-extendibility. For the converse, suppose that  $\Sigma$  is a set of strings for which  $G(\Sigma)$  has no obstructions.

We start by reducing to the case where the point set  $\bigcup \Sigma$  is connected: iteratively add a new string in a face of  $\bigcup \Sigma$  connecting two connected components of  $\bigcup \Sigma$ . No obstruction is introduced at each step (obstructions are cycles), and, eventually, the obtained set  $\bigcup \Sigma$  is connected. An extension of the new set of strings contains an extension for the original set, thus we may assume that  $\bigcup \Sigma$  is connected.

Our proof is algorithmic, and consists of repeatedly applying one of the three steps described below.

- **Disentangling Step.** If a string  $\sigma \in \Sigma$  has an end  $a$  with degree at least 2 in  $G(\Sigma)$ , then we slightly extend the  $a$ -end of  $\sigma$  into one of the faces incident with  $a$ .
- **Face-Escaping Step.** If a string  $\sigma \in \Sigma$  has an end  $a$  with degree 1 in  $G(\Sigma)$ , and is incident with an interior face, then we extend the  $a$ -end of  $\sigma$  until it intersects some point in the boundary of this face.
- **Exterior-Meeting Step.** Assuming that all the strings in  $\Sigma$  have their two ends in the outer face and these ends have degree 1 in  $G(\Sigma)$ , we extend the ends of two disjoint strings so that they meet in the outer face.

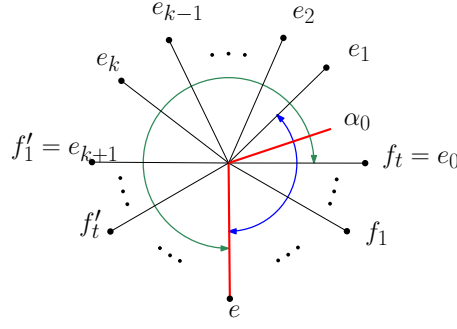
Each of these three steps either increases the number of pairs of strings that intersect, or increase the number crossings (recall that a crossing between  $\sigma$  and  $\sigma'$  is a non-tangential intersection point in  $\sigma \cap \sigma'$  that is not an end of  $\sigma$  or  $\sigma'$ ). Moreover, these steps can be performed as long as one of the next two conditions holds: (1) at least one string does not have an end incident with the outer face; and (2) there is a pair of strings that do not cross. If none of (1) and (2) hold, then our set of strings is extendible into an arrangement of pseudolines. Henceforth, we will show that, if performed correctly, none of these steps introduces an obstruction. The proof for each step can be read independently.

► **Lemma 7 (Disentangling Step).** *Suppose that  $\sigma \in \Sigma$  has an end  $a$  with degree at least 2 in  $G(\Sigma)$ . Then we can extend the  $a$ -end of  $\sigma$  into one of the faces incident to  $a$  without creating an obstruction.*

**Proof.** A pair of different edges  $f$  and  $f'$  in  $G(\Sigma)$  incident with  $a$  are *twins* if they belong to the same string in  $\Sigma$ . The edge  $e \subseteq \sigma$  incident with  $a$  has no twin.

The fact that no pair of strings tangentially intersect at  $a$  tells us that if  $(f_1, f'_1)$  and  $(f_2, f'_2)$  are pairs of twins, then  $f_1, f_2, f'_1, f'_2$  occur in this cyclic order for either the clockwise or counterclockwise rotation at  $a$ . Thus, we may assume that the counterclockwise rotation at  $a$  restricted to the twins and  $e$  is  $e, f_1, \dots, f_t, f'_1, \dots, f'_t$ , where  $(f_i, f'_i)$  is a twin pair for  $i = 1, \dots, t$ .

To avoid tangential intersections, the extension of  $\sigma$  at  $a$  must be in the angle between  $f_t$  and  $f'_1$  not containing  $e$ . Let  $e_1, \dots, e_k$  be the counterclockwise ordered list of non-twin edges at  $a$  having an end in this angle (as depicted in Figure 4). We label  $e_0 = f_t$  and  $e_{k+1} = f'_1$ . If there are no twins, then let  $e_0 = e_{k+1} = e$ .



■ **Figure 4** Substrings included in the disk bounded by  $C_0$ .

Let us consider all the possible extensions: for  $i \in \{0, \dots, k\}$ , let  $\Sigma_i$  be the set of strings obtained from  $\Sigma$  by slightly extending the  $a$ -end of  $\sigma$  into the face containing the angle between  $e_i$  and  $e_{i+1}$ . Let  $\alpha_i$  be the new edge at  $a$  extending  $\sigma$  in  $\Sigma_i$  (see  $\alpha_0$  in Figure 4).

Seeking a contradiction, suppose that, for each  $i \in \{0, \dots, k\}$ ,  $G(\Sigma_i)$  contains an obstruction  $C_i$ . Since  $\alpha_i$  contains a degree-1 vertex,  $\alpha_i$  is not in  $C_i$ . Hence  $C_i$  is a cycle of  $G(\Sigma)$ . Thus,  $C_i$  is not an obstruction in  $G(\Sigma)$  and becomes an obstruction in  $G(\Sigma_i)$ . This conversion has a simple explanation: in  $G(\Sigma)$ ,  $C_i$  has exactly three rainbows, and one of them is  $a$ . After  $\alpha_i$  is added,  $a$  is not a rainbow in  $C_i$  (witnessed by the edges  $e$  and  $\alpha_i$  included in the new version of  $\sigma$ ).

Recall from Section 3 that a *near-obstruction at  $a$*  is a cycle with exactly three rainbows, and one of them is  $a$ . Each of  $C_0, C_1, \dots, C_k$  is a near-obstruction at  $a$  in  $G(\Sigma)$ .

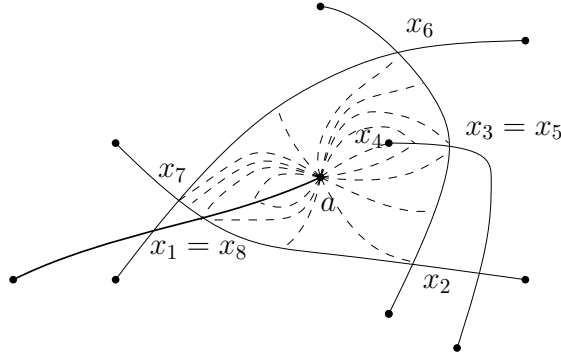
For a cycle  $C \subseteq G$ , let  $\Delta(C)$  denote the closed disk bounded by  $C$ . Both  $e$  and  $\alpha_0$  are in  $\Delta(C_0)$ . Thus, either  $\Delta(C_0) \supseteq \{e, f_1, f_2, \dots, f_t, e_1\}$  (blue bidirectional arrow in Figure 4) or  $\Delta(C_0) \supseteq \{f_t, e_1, \dots, e_k, f'_1, f'_2, \dots, f'_t, e\}$  (green bidirectional arrow). We rule out the latter situation as the second list contains  $f_t$  and  $f'_t$ , and this would imply that  $a$  is not a rainbow for  $C_0$  in  $G(\Sigma)$ .

We just showed that  $\{e, e_0, e_1\} \subseteq \Delta(C_0)$ . By symmetry,  $\{e_k, e_{k+1}, e\} \subseteq \Delta(C_k)$ . Consider the largest index  $i \in \{0, 1, \dots, k-1\}$  for which  $\{e, e_0, \dots, e_{i+1}\} \subseteq \Delta(C_i)$ . By the choice of  $i$ , and because  $\{e, \alpha_{i+1}\} \subseteq \Delta(C_{i+1})$ ,  $\{e, f'_1, \dots, f'_i, e_k, \dots, e_i\} \subseteq \Delta(C_{i+1})$ . Apply Lemma 5 to the pair  $C_i$  and  $C_{i+1}$ , where  $C_i, C_{i+1}$  and  $a$  play the roles of  $C_1, C_2$  and  $v$ . Condition 1 of Lemma 5 holds, and hence we obtain that  $G(\Sigma)$  has an obstruction, a contradiction. ◀

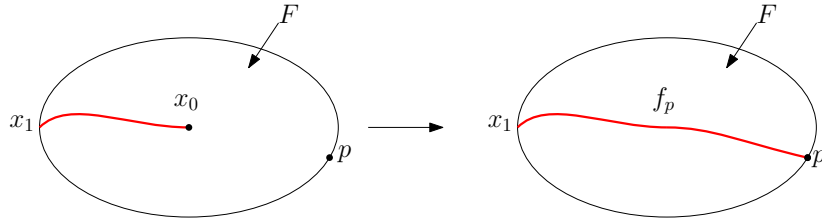
► **Lemma 8 (Face-Escaping Step).** *Suppose that there is a string  $\sigma$  that has an end  $a$  with degree 1 in  $G(\Sigma)$ , and  $a$  is incident to an interior face  $F$ . Then there is an extension  $\sigma'$  of  $\sigma$  from its  $a$ -end to a point in the boundary of  $F$  such that the set  $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$  has no obstruction.*

**Proof.** Let  $W$  be the closed boundary walk  $(x_0, e_1, x_1, e_2, \dots, e_n, x_n)$  of  $F$  such that  $x_0 = x_n = a$  and  $F$  is to the left as we traverse  $W$  (see Figure 5 for an illustration with  $n = 9$ ). For  $i = 1, \dots, n$  we let  $m_i$  be a point in the relative interior of  $e_i$ , and let  $P$  be the list of non-necessarily distinct points  $m_1, x_1, m_2, x_2, \dots, m_n$ , which are the potential ends for all the different extensions. For each  $p \in P$ , let  $\Sigma_p$  be the set of strings obtained from  $\Sigma$  by extending the  $a$ -end of  $\sigma$  by adding an arc  $\alpha_p$  connecting  $a$  to  $p$  in  $F$  (see Figure 5). We assume that every two distinct arcs  $\alpha_p$  and  $\alpha_{p'}$  are internally disjoint.

Let  $f_p$  be the edge  $e_1 \cup \alpha_p$  in  $G(\Sigma_p)$ ;  $f_p$  has ends  $x_1$  and  $p$ . Also, let  $\sigma^p = \sigma \cup \alpha_p$ . See Figure 6. Seeking a contradiction, suppose that each  $G(\Sigma_p)$  has an obstruction.



■ **Figure 5** All possible extensions in the Face-Escaping Step.



■ **Figure 6** Transforming  $\Sigma$  into  $\Sigma_p$ .

358  $\triangleright$  **Claim 9.** Let  $p \in P$ . Then there exists an obstruction  $C_p$  in  $G(\Sigma_p)$  including  $f_p$ . Moreover,

- 359 (1) if  $p \in \sigma$ , then  $C_p$  can be chosen so that all its edges are included in  $\sigma^p$ ; and  
 360 (2) if  $p \notin \sigma$ , then every obstruction includes  $f_p$ .

361 **Proof.** First, if  $p \in \sigma$ , then the string  $\sigma^p$  self-intersects at  $p$ ; thus  $\sigma^p$  has a simple close curve  
 362 including  $f_p$ . In this case let  $C_p$  be the cycle in  $G(\Sigma_p)$  representing this simple closed curve  
 363 without rainbows, and thus (1) holds.

364 Second, assume that  $p \notin \sigma$  and let  $C_p$  be any obstruction of  $G(\Sigma_p)$ . For (2), we will show  
 365 that  $f_p \in E(C_p)$ .

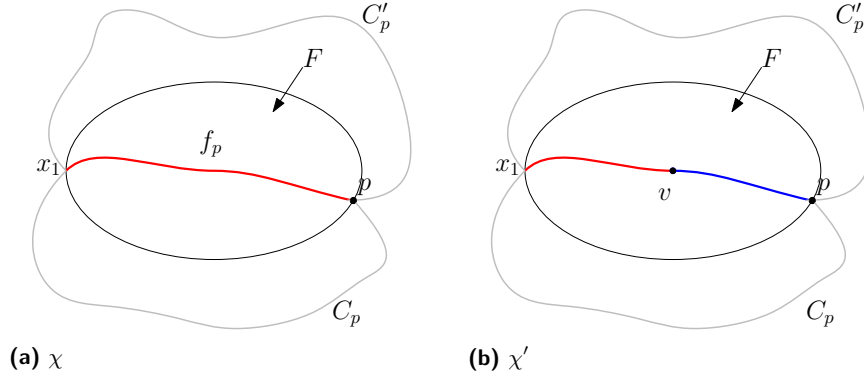
366 Seeking a contradiction, suppose that  $f_p \notin E(C_p)$ .

367 If  $p = m_i$  for  $i \in \{1, \dots, n\}$ , since  $m_i$  is the only vertex whose rotation in  $G(\Sigma)$  differs  
 368 from its rotation in  $G(\Sigma_{m_i})$ ,  $m_i \in V(C_p)$ . Consider the cycle  $C$  of  $G(\Sigma)$  obtained from  $C_p$   
 369 by replacing the subpath  $(x_{i-1}, m_i, x_i)$  by the edge  $x_{i-1}x_i$ . For each vertex  $v \in V(C)$  the  
 370 colors of the edges of  $G(\Sigma)$  at  $v$  included in the disk bounded by  $C$  are the same as in  $G(\Sigma_p)$   
 371 for the disk bounded by  $V(C_p)$ . Thus,  $C$  is an obstruction for  $G(\Sigma)$ , a contradiction.

372 Suppose now that  $p$  is one of  $x_1, \dots, x_{n-1}$ . The only vertex in  $G(\Sigma)$  whose rotation is  
 373 different in  $G(\Sigma_p)$  is  $p$ . Therefore,  $p$  is a point that is a rainbow for  $C_p$  in  $G(\Sigma)$ , but not a  
 374 rainbow in  $G(\Sigma_p)$ , as witnessed by the two edges of  $\sigma^p$  that are incident with  $p$  and inside  
 375  $C_p$ . This contradicts the assumption that  $p \notin \sigma$ . Hence  $f_p \in E(C_p)$ .  $\blacktriangleleft$

376 Henceforth we assume that, for  $p \in P$ ,  $C_p$  is an obstruction in  $G(\Sigma_p)$  as in Claim 9.

377 More can be said about the obstructions in  $G(\Sigma_p)$ , but for this we need some terminology.  
 378 If we orient an edge  $e$  in a plane graph, then the *sides* of  $e$  are either the points near  $e$  that  
 379 are to the right of  $e$ , or the points near  $e$  to the left of  $e$ . For any cycle  $C$  of  $G$  through  $e$ ,  
 380 exactly one side of  $e$  lies inside  $C$ . This is the side of  $e$  *covered* by  $C$ . For the next claim  
 381 and in the rest of the proof we will assume that for  $p \in P$ ,  $f_p$  is oriented from  $x_1$  to  $p$ .



■ **Figure 7** The two edge colorings  $\chi$  and  $\chi'$  discussed in the proof of Claim 10.

382 ▷ **Claim 10.** For  $p \in P$  with  $p \notin \sigma$ , every obstruction in  $G(\Sigma_p)$  covers the same side of  $f_p$ .

383 **Proof.** Suppose that for  $p \in P$  there are obstructions  $C_p$  and  $C'_p$  covering both sides of  $f_p$ .  
 384 Let  $G'$  be the plane graph obtained from  $G(\Sigma_p)$  by subdividing  $f_p$ , and let  $v$  be the new  
 385 degree-2 vertex inside  $f_p$ .

386 We consider the edge-colouring  $\chi$  induced by the strings in  $\Sigma_p$ . Let  $\chi'$  be a new colouring  
 387 obtained from  $\chi$  by replacing the colour of the edge  $vp$  by a new colour not used in  $\chi$   
 388 (see Figure 7). It is immediate that (i)  $\chi'$  induces a path-partition in  $G'$ ; and in the next  
 389 paragraph we show that (ii)  $C_p$  and  $C'_p$  are near-obstructions for  $v$  with respect to  $\chi'$ .

390 Consider the set of edges in the rotation at  $p$  inside the disk bounded by  $C_p$  and assume  
 391 they are colored by  $\chi$ . No edge from this set (except  $f_p$ ) can have the same color as  $f_p$  or else  
 392  $p \in \sigma$ , contradicting our hypothesis. Therefore,  $p$  is a rainbow in  $C_p$  in  $\chi$  if and only if  $p$  is a  
 393 rainbow in  $C_p$  in  $\chi'$ . Thus, when we switch from  $\chi$  to  $\chi'$ ,  $v$  is the only vertex of  $C_p$  switching  
 394 identity (where the identity is to be or not to be a rainbow). As  $C_p$  is an obstruction for  $\chi$ ,  
 395 then  $C_p$  is a near obstruction at  $v$  for  $\chi'$ . Likewise,  $C'_p$  is a near obstruction for  $\chi'$ .

396 As Condition 2 of Lemma 5 holds for  $C_1 = C_p$ ,  $C_2 = C'_p$  and  $v = v$  with respect to  
 397  $\chi'$ ,  $G'$  has an obstruction not containing  $v$  in  $\chi'$ . However, this implies the existence of an  
 398 obstruction in  $G(\Sigma)$  with respect to  $\chi$ , a contradiction. ◀

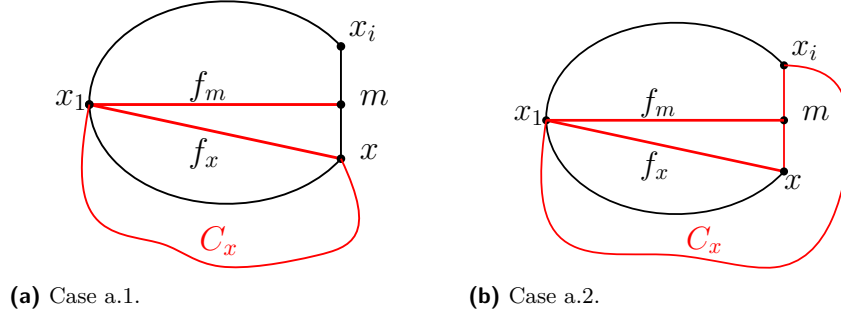
399 Recall that the boundary walk of  $F$  is  $W = (x_0, e_1, \dots, e_n, x_n)$ , with  $x_0 = x_n = a$ . Since  
 400  $x_1$  and  $x_{n-1}$  are in  $\sigma$ , the extreme obstructions  $C_{x_1}$  and  $C_{x_2}$  cover the right of  $f_{x_1}$  and the  
 401 left of  $f_{x_{n-1}}$ , respectively. Thus, there are two consecutive vertices  $x_{i-1}, x_i$  in  $W - a$ , such  
 402 that the interior of  $C_{x_{i-1}}$  covers the right of  $f_{x_{i-1}}$  and the interior of  $C_{x_i}$  covers the left of  
 403  $f_{x_i}$ . Moreover, we may assume that the interior of  $C_{m_i}$  includes the left of  $f_{m_i}$  (otherwise  
 404 we reflect our drawing).

405 The next claim is the last ingredient to obtain a final contradiction. To make the notation  
 406 simpler, we let  $x = x_{i-1}$  and  $m = m_i$ .

407 ▷ **Claim 11.** Exactly one of the following holds:

- 408 (a)  $x \in \sigma$ ,  $m \notin \sigma$  and  $G(\Sigma_m)$  has an obstruction covering the side of  $f_m$  not covered by  $C_m$ ;  
 409 or  
 410 (b)  $x \notin \sigma$  and  $G(\Sigma_x)$  has an obstruction covering the side of  $f_x$  not covered by  $C_x$ .

411 **Proof.** By redrawing the arcs representing  $f_x$  and  $f_m$ , we will assume that they only intersect  
 412 at  $x_1$ . In particular this redrawing creates two copies of the edge  $e_1$ .



■ **Figure 8** Illustrations for Claim 11.a.

413 First, suppose that  $x \in \sigma$ . For (a) we have two cases depending on whether  $xx_i$  is an  
 414 edge in  $C_x$ .

415 **Case a.1**  $xx_i \notin E(C_x)$ . See Figure 8a.

416 Let  $C'_m$  be the cycle obtained from  $C_x$  by replacing the edge  $f_x$  by the path  $P = (x_1,$   
 417  $f_m, m, mx, x)$ . Since  $x \in \sigma$ , by the choice of  $C_x$  (Claim 9), all the edges in  $C_x$  are in  $\sigma^x$ .  
 418 Therefore, by Claim 9.1, all the edges in  $C'_m$ , with the possible exception of  $mx$ , are in  $\sigma^m$ .  
 419 Thus  $C'_m$  is an obstruction in  $G(\Sigma_m)$ .

420 Now we show that  $C'_m$  covers the right side of  $f_m$ . The disk bounded by  $P \cup f_x$  is to the  
 421 right of  $f_m$  as this side of  $P \cup f_x$  is included in the bounded face  $F$ . Since the interior of  $C_x$   
 422 is to the right of  $f_x$ , the interior of  $C'_m$  covers the right side of  $f_m$ .

423 Finally, note that  $m \notin \sigma$ , or else,  $C'_m \subseteq \sigma^m$  and hence by the choice of  $C_m$ , and Claim  
 424 10,  $C'_m = C_m$ . However, this contradicts that  $C_m$  covers the left side of  $f_m$ . Thus, (a) holds.

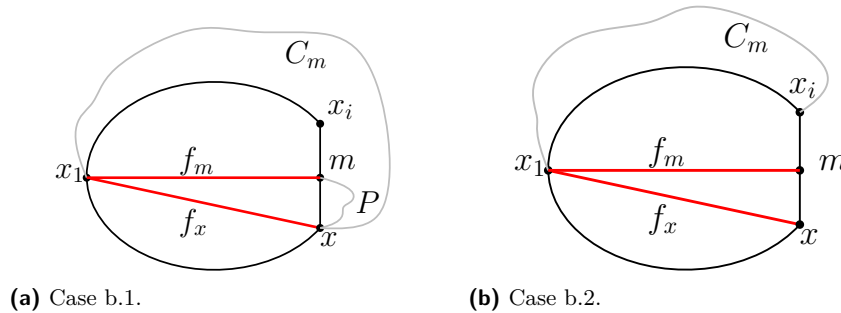
425 **Case a.2.**  $xx_i \in E(C_x)$ . See Figure 8b.

426 Let  $C'_m$  be the cycle obtained from  $C_x$  by replacing the path  $(x_1, f_x, x, xx_i, x_i)$  by  $(x_1, f_m, m,$   
 427  $mx_i, x_i)$ . Since  $x \in \sigma$ , by the choice of  $C_x$  (Claim 9), all the edges in  $C_x$  are in  $\sigma^x$ . Therefore  
 428 all the edges in  $C'_m$  are in  $\sigma^m$ . Thus  $C'_m$  is an obstruction in  $G(\Sigma_m)$ .

429 Now we show that  $C'_m$  covers the right side of  $f_m$ . The disk bounded by  $f_x \cup f_m \cup xm$  is  
 430 to the right of  $f_m$  as this side of  $f_x \cup f_m \cup xm$  is included in the bounded face  $F$ . Since the  
 431 interior of  $C_x$  is to the right of  $f_x$ , the interior of  $C'_m$  covers the right side of  $f_m$ .

432 Finally, as  $C'_m \subseteq \sigma^m$  and by the choice of  $C_m$ ,  $C'_m = C_m$ . However, this contradicts the  
 433 assumption that  $C_m$  covers the left side of  $f_m$ . Thus, (a) holds.

434 Turning to (b), let us suppose that  $x \notin \sigma$ .



■ **Figure 9** Illustrations for Claim 11.b.

435 **Case b.1.**  $x \in V(C_m)$ . See Figure 9a.

436 Let  $T$  be the triangle bounded by  $f_x$ ,  $f_m$  and  $xm$ . The interior face of  $T$  is to the left of  
 437  $f_x$  and to the right of  $f_m$ . Let  $P$  be the  $mx$ -path of  $C_m - f_m$  and let  $P'$  be the  $xx_1$ -path of  
 438  $C_m - m$ . Since the interior face of  $T$  is a subset of  $F$ ,  $P$  and  $P'$  are drawn in the closure of  
 439 the exterior of  $T$  (possibly  $P = (m, mx, x)$ ).

440 Let  $C$  be the simple closed curve bounded by  $P \cup f_x \cup f_m$  (in other words,  $C$  is obtained  
 441 from  $T$  by replacing  $xm$  by  $P$ ). Seeking a contradiction, suppose that  $xm$  is in the closed  
 442 exterior of  $C$ . Then,  $P'$  is included inside the cycle  $C' = P + xm$ . Since  $V(C') \subseteq V(C_m)$   
 443 and  $C_m$  is included in the disk bounded by  $C'$ , the number of rainbows in  $C'$  is at most  
 444 the number of rainbows in  $C_m$ . Then  $C'$  is an obstruction in  $G(\Sigma_m)$  not containing  $f_m$ ,  
 445 contradicting Claim 9.2. Thus,  $xm$  is inside  $C$ .

446 Our last observation implies that  $P'$  is an arc connecting  $x_1$  and  $x$  in the exterior of  $C$ .  
 447 Since the interior of  $C_m$  covers the left of  $f_m$ , the interior of  $C'_x = P' + f_x$  covers the left of  
 448  $f_x$ . The cycle  $C'_x$  is an obstruction because  $V(C'_x) \subseteq V(C_m)$  and  $C_m$  is included inside  $C'_x$ .

449 **Case b.2.**  $x \notin V(C_m)$ . See Figure 9b.

450 In this case we let  $C'_x$  be the cycle obtained by replacing the path  $(x_1, f_m, m, mx_i, x_i)$  in  
 451  $C_m$  by the path  $P = (x_1, f_x, x, xx_i, x_i)$  in  $G(\Sigma_x)$ . Since  $C_m$  covers the left of  $f_m$  and  $F$  is  
 452 bounded,  $C'_x$  covers the left of  $f_x$ .

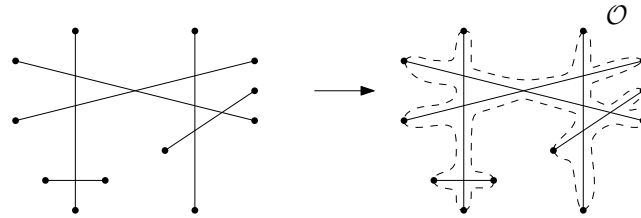
453 To show that  $C'_x$  is an obstruction, note that  $C_m$  is inside  $C'_x$  and that  $V(C'_x) \setminus \{x\} \subseteq$   
 454  $V(C_m)$ . Thus, all the rainbows of  $C'_x$  in  $V(C'_x) \setminus \{x\}$  are also rainbows in  $C_m$ . Since  $x \notin \sigma$ ,  
 455 we see that  $x$  is a rainbow in  $C'_x$ , but is not a vertex of  $C_m$ . To compensate, we note  
 456 that  $m$  is a rainbow in  $C_m$  that is not in  $V(C'_x)$ : if  $m$  is not rainbow, both  $f_m$  and  $xx_i$  are  
 457 included in  $\sigma$ , implying that  $x \in \sigma$ . This shows that  $C'_x$  has at most as many rainbows as  
 458  $C_m$ . Therefore  $C'_x$  is the desired obstruction. ◀

459 Claims 10 and 11 contradict each other, so, for some  $p \in P$ ,  $G(\Sigma_p)$  has no obstructions.

460 ◀

461 ► **Lemma 12** (Exterior-Meeting Step). *If all the strings in  $\Sigma$  have their ends on the outer*  
 462 *face of  $G(\Sigma)$  and the ends have degree 1 in  $G(\Sigma)$ , then we can extend a pair disjoint strings*  
 463 *so that they intersect without creating an obstruction.*

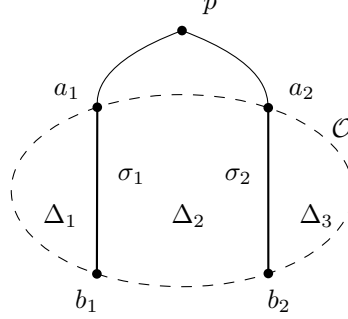
464 **Proof.** First, consider a simple closed curve in the outerface of  $\bigcup \Sigma$  closely following the  
 465 outerboundary of  $\bigcup \Sigma$ . Then, by slightly modifying this curve, we obtain a simple closed  
 466 curve  $\mathcal{O}$  containing all the ends of the strings in  $\Sigma$ , but otherwise disjoint from  $\bigcup \Sigma$ . See  
 467 Figure 10.



■ **Figure 10** Construction of the curve  $\mathcal{O}$ .

468 Suppose  $\sigma_1, \sigma_2$  are two disjoint strings in  $\Sigma$ . For  $i = 1, 2$ , let  $a_i, b_i$  be the ends of  $\sigma_i$ ;  
 469 since  $\sigma_1$  and  $\sigma_2$  do not cross, we may assume that these ends occur in the cyclic order  $a_1, b_1$ ,

470  $b_2, a_2$ . We extend the  $a_i$ -ends of  $\sigma_1$  and  $\sigma_2$  so that they meet in a point  $p$  in the outer face,  
 471 and so that all the ends of  $\sigma_1$  and  $\sigma_2$  remain incident with the outer face (Figure 11). Let  $\Sigma'$   
 472 be the obtained set of strings.



■ **Figure 11** Exterior-Meeting Step.

473 Seeking a contradiction, suppose that  $G(\Sigma')$  has an obstruction  $C$ . Since  $G(\Sigma)$  has no  
 474 obstruction,  $p \in V(C)$ . Our contradiction will be to find three rainbows in  $C$ . Note that  
 475  $p$  is a rainbow. To obtain a second rainbow, traverse  $C$  starting from  $p$  towards  $a_1$ . Let  
 476  $d_1$  be the first vertex during our traversal that is not in the extended  $\sigma_1$ , and let  $c_1$  be its  
 477 neighbour in  $\sigma_1$ , one step before we reach  $d_1$ . Since  $b_1$  has degree one,  $c_1 \neq b_1$ .

478 The strings  $\sigma_1$  and  $\sigma_2$  divide the disk bounded by  $\mathcal{O}$  into three closed regions  $\Delta_1, \Delta_2,$   
 479  $\Delta_3$  such that  $\Delta_1 \cap \Delta_2 = \sigma_1$ ,  $\Delta_2 \cap \Delta_3 = \sigma_2$  and  $\Delta_1 \cap \Delta_3 = \emptyset$  (see Figure 11).

480 **Claim 13.** The cycle  $C$  has a rainbow included in  $\Delta_1$ .

481 **Proof.** First, suppose that  $d_1 \notin \Delta_1$ . In this case,  $c_1$  is a rainbow because otherwise there  
 482 would be a string  $\sigma$  that tangentially intersects  $\sigma_1$  at  $c_1$ . Thus, if  $d_1 \notin \Delta_1$ , then  $c_1$  is the  
 483 desired rainbow.

484 Second, suppose that  $d_1 \in \Delta_1$ . Let  $P_1$  be the path of  $C$  starting at  $c_1$ , continuing on the  
 485 edge  $c_1 d_1$ , and ending at the first vertex we encounter in  $\sigma_1$ . Let  $C'$  be the cycle enclosed by  
 486  $P_1 \cup \sigma_1$ . Since  $C'$  is not an obstruction, there is one rainbow of  $C'$  that is an interior vertex  
 487 of  $P_1$ ; this is the desired rainbow of  $C$ . This concludes the proof of Claim 13. ◀

488 Considering  $\sigma_2$  instead of  $\sigma_1$ , Claim 13 yields a third rainbow in  $C$  inside the region  $\Delta_3$   
 489 analogous to  $\Delta_1$ , contradicting that  $C$  is an obstruction. Hence Lemma 12 holds. ◀

490 We iteratively apply the Disentangling Step, Face-Escaping Step or Exterior-Meeting Step  
 491 without creating obstructions. Each step increases the number of pairwise intersecting strings  
 492 in  $\Sigma$  until we reach a stage where the strings are pairwise intersecting and all of them have  
 493 their two ends in the unbounded face. From this we extend them into an arrangement of  
 494 pseudolines. This concludes the proof of Theorem 2. ◀

## 495 **5 Finding obstructions and extending strings in polynomial time**

496 We start this section by describing an algorithm to detect obstructions. Henceforth, we  
 497 assume that the input to the problem is the planarization  $G(\Sigma)$  of an ordinary set of  $s$  strings  
 498  $\Sigma$ . For the running-time analysis, we assume that  $n$  and  $m$  are the number of vertices and  
 499 edges in  $G(\Sigma)$ , respectively. Since  $G(\Sigma)$  is planar,  $m = O(n)$ . Moreover, if  $\Sigma$  is pseudolinear,  
 500 then  $n \leq \binom{s}{2} + 2s = \binom{s+2}{2} - 1$ . At the end of this section we explain how to extend  $\Sigma$  (if  
 501 possible) in polynomial time.



Recall that each string in  $\Sigma$  receives a different colour; this induces an edge-colouring on  $G(\Sigma)$  where each string is a monochromatic path. An *outer-rainbow* is a vertex  $x \in V(G(\Sigma))$  incident with the outer face and for which the edges incident with  $x$  have different colours. Next we describe the basic operation in our obstruction-detecting algorithm.

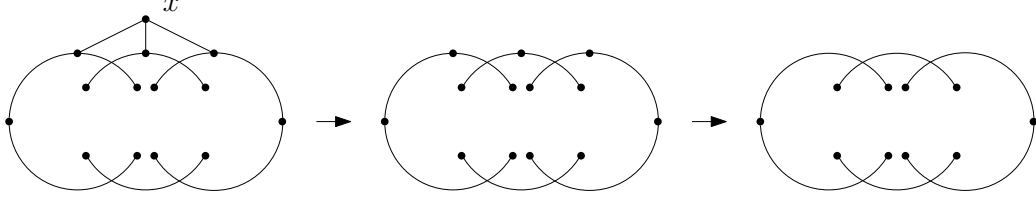


Figure 12 From  $\Sigma$  to  $\Sigma - x$ .

**Outer-rainbow deletion.** Given an outer-rainbow  $x \in V(G(\Sigma))$ , the instance  $G(\Sigma - x)$  is defined by: first, removing  $x$  and the edges incident to  $x$ ; second, suppressing the degree-2 vertices incident with edges of the same colour; and third, removing remaining degree-0 vertices (Figure 12 illustrates this process). Edge colours are preserved.

It is easy to verify that  $G(\Sigma - x)$  is the planarization of an arrangement of strings. The colours removed by this operation are those belonging to strings that are paths of length 1 in  $G(\Sigma)$  incident with  $x$ . Our obstruction-detecting algorithm relies on the following property:

(\*\*) if  $x$  is an outer-rainbow of  $G(\Sigma)$ , then there is an obstruction in  $G(\Sigma)$  not including  $x$  if and only if there is an obstruction in  $G(\Sigma - x)$ .

This property holds because cycles in  $G(\Sigma) - x$  and in  $G(\Sigma - x)$  are in 1-1 correspondence: two cycles correspond to each other if they are the same simple closed curve. This correspondence is obstruction-preserving.

Let us now describe the two subroutines in our algorithm. For this, we remark that an outer-rainbow of  $G(\Sigma)$  is a rainbow for any cycle containing it.

**Subroutine 1.** *Detecting an obstruction through two outer-rainbows  $x$  and  $y$ .*

- (1) Find a cycle  $C$  through  $x$  and  $y$  whose edges are incident with the outer face of  $G(\Sigma)$ . If  $C$  exists, then this cycle is unique and can be described as the outer boundary of the block containing  $x$  and  $y$ . If no such  $C$  exists, then output *No obstruction through  $x$  and  $y$* . Else, go to Step 2.
- (2) Find whether there is a third outer-rainbow  $z \in V(C) \setminus \{x, y\}$ . If such  $z$  exists, update  $G(\Sigma) \leftarrow G(\Sigma - z)$  and go to Step 1. If no such  $z$  exists, output  $C$ .

*Correctness and running-time of Subroutine 1:* If an obstruction through  $x$  and  $y$  exists, then  $x$  and  $y$  are in the same block (some authors use the term ‘biconnected component’). Since  $x$  and  $y$  are incident with the outer face, the outer boundary of the block containing  $x$  and  $y$  is the cycle  $C$  from Step 1. This  $C$  can be found by considering outer boundary walk  $W$  of  $G(\Sigma)$  and then by finding whether  $x$  and  $y$  belong to the same non-edge block of  $W$ . Finding  $W$  is  $O(m)$  and computing the blocks of  $W$  via a DFS takes  $O(m)$  time.

In Step 2, if there is a third outer rainbow  $z$  in  $C$ , then no obstruction through  $x$  and  $y$  contains  $z$ . Property (\*\*) justifies the update that takes  $O(m)$  time.

A full run from Step 1 to Step 2 takes  $O(m)$ . Moving from Step 2 to Step 1 occurs  $O(n)$  times. Thus, the total time for Subroutine 1 is  $O(mn) = O(n^2)$ .

**Subroutine 2.** *Detecting an obstruction through a single outer-rainbow  $x$ .*

- 538 (1) Find a cycle  $C$  through  $x$  whose edges are incident with the outer face of  $G(\Sigma)$ . If no  
 539 such  $C$  exists, output *No obstruction through  $x$* . Else, go to Step 2.  
 540 (2) Find whether there is an outer-rainbow  $y$  in  $V(C) \setminus \{x\}$ . If no such  $y$  exists, output  $C$ .  
 541 Else, apply Subroutine 1 to  $x$  and  $y$ ; if there is an obstruction  $C'$  through  $x$  and  $y$ , then  
 542 output  $C'$ . Else, update  $G(\Sigma) \leftarrow G(\Sigma - y)$  and go to Step 1.

543 *Correctness and running-time of Subroutine 2:* If  $G(\Sigma)$  has an obstruction through  $x$ , then  
 544 there is a non-edge block in  $G(\Sigma)$  containing  $x$ . The outer boundary of this block is a cycle  
 545  $C$  through  $x$  having all edges incident with the outer face. As in Subroutine 1, Step 1 takes  
 546  $O(m)$  time.

547 Detecting the existence of  $y$  in Step 2 is  $O(m)$  because to detect rainbows in  $C$ , each edge  
 548 incident with a vertex in  $V(C)$  is verified at most twice. The update in Step 2 is justified  
 549 by Property (\*\*). Since Step 2 may use Subroutine 1, Step 2 takes  $O(n^2)$  time. As moving  
 550 from Step 2 to Step 1 occurs  $O(n)$  times, the total running-time for Subroutine 2 is  $O(n^3)$ .

551 We are now ready for the algorithm to detect obstructions.

552 **Algorithm 1:** *Detecting obstructions in  $G(\Sigma)$ .*

- 553 (1) Find a cycle  $C$  having all edges incident with the outer face. If no such  $C$  exists, output  
 554 *No obstruction*. Else, go to step 2.  
 555 (2) Find whether there is an outer rainbow  $x \in V(C)$ . If not, output  $C$ . Else apply Subroutine  
 556 2 to  $x$ ; if there is an obstruction  $C'$  through  $x$ , output  $C'$ . Else, update  $G(\Sigma) \leftarrow G(\Sigma - x)$   
 557 and go to Step 1.

558 *Correctness and running-time of Algorithm 1:* If  $G(\Sigma)$  has an obstruction, then it has a  
 559 non-trivial block whose outer boundary is a cycle  $C$  as in Step 1. As before,  $C$  and  $x$  as in  
 560 Step 2 can be found in  $O(m)$  steps. If  $C$  has not outer rainbow  $x$ , then  $C$  is an obstruction;  
 561 Property (\*\*) justifies the update in Step 2.

562 Since Step 2 may use Subroutine 2, a full run of Steps 1 and 2 takes  $O(n^3)$  time. Since  
 563 Step 2 goes to Step 1  $O(n)$  times, the running-time of Algorithm 1 is  $O(n^4)$ .

564 Algorithm 1 and the constructive proof of Theorem 2 imply the following result.

565 ► **Theorem 14.** *There is a polynomial-time algorithm to recognize and extend an ordinary  
 566 set of strings that are extendible to an arrangement of pseudolines.*

567 **Proof.** Let  $\Sigma$  be an ordinary set of  $s$  strings. First, note that if  $n = |V(G(\Sigma))|$ ,  $m =$   
 568  $|E(G(\Sigma))|$ , and  $\Sigma$  is extendible, then  $n \leq \binom{s}{2} + 2s$ . Hence  $n, m = O(s^2)$ .

569 Assume that  $G(\Sigma)$  has not obstructions, by first verifying that  $n \leq \binom{s}{2} + 2s$  and then  
 570 running Algorithm 1. For each end in each string in  $\Sigma$ , we keep track of whether one of the  
 571 Disentangling, Face-Escaping or Exterior-Meeting Steps apply.

572 The Disentangling and Face-Escaping Steps consist on extending one end  $a$  of a fixed  
 573 string  $\sigma \in \Sigma$  in different ways to find an obstruction-free set of strings. For the Disentangling  
 574 Step, the number of possible extensions is bounded by the maximum degree of  $G(\Sigma)$ ; for the  
 575 Face-Escaping Step, the number of possible extensions is bounded by twice the length of the  
 576 face containing the end that we are extending. Thus, each step lead to  $O(m)$  possibilities, and  
 577 testing obstructions in each of them is  $O(n^4)$ . Thus, the Disentangling and the Face-Escaping  
 578 Steps take  $O(n^5)$  time.

579 The Exterior-Meeting Step is  $O(m^2)$  because for this step we just need to record the  
 580 number of the pairwise disjoint strings in  $\Sigma$  and the set of strings that have ends incident  
 581 with the outer face; if all the strings have their ends in the outer boundary, the extension is  
 582 performed as in the proof of Lemma 12.

583 As there is a total of  $O(s^2)$  extending steps, extending  $\Sigma$  is  $O(s^2(n^5 + m^2)) = O(s^{12})$ . ◀

## 6 Concluding remarks

In this work we characterized in Theorem 2 sets of strings that can be extended into arrangements of pseudolines. Moreover, we showed that the obstructions to pseudolinearity can be detected in  $O(n^4)$  time, where  $n$  is the number of vertices in the planarization of the set of strings.

An easy consequence of Theorem 2 is the following (presented before as Theorem 1).

► **Theorem 15.** *Let  $D$  be a non-pseudolinear good drawing of a graph  $H$ . Then there is a subset  $S$  of edge-arcs in  $\{D[e] : e \in E(H)\}$ , such that each  $\sigma \in S$  has a substring  $\sigma' \subseteq \sigma$  for which  $\bigcup_{\sigma \in S} \sigma'$  is one of the drawings represented in Figure 2.*

**Proof.** Take  $C$  an obstruction of the planarization associated to  $D$ . Let  $\delta(C) \subseteq V(C)$  be the vertices that in  $C$  are incident with two different strings in  $\Sigma = \{D[e] : e \in E(H)\}$ . We choose our obstruction  $C$  so that  $|\delta(C)|$  is as small as possible.

Decompose  $C$  into a cyclic sequence of paths  $P_0, \dots, P_m$ , where  $P_i$  connects two points in  $\delta(C)$  and it is otherwise disjoint from  $\delta(C)$ . Using Lemma 4, one can show that  $P_0, \dots, P_m$  belong to distinct edge-arcs  $\sigma_0, \dots, \sigma_m \in \Sigma$ , respectively. For each  $P_i$ , we consider the string  $\sigma'_i$ , obtained by slightly extending the ends of  $P_i$  that are not rainbows in  $C$ ; we extend them along  $\sigma_i$ .

Let  $x \in \delta(C)$  be an end shared by  $P_{i-1}$  and  $P_i$ . If  $x$  is not a rainbow for  $C$ , then  $x$  is a crossing between  $\sigma_{i-1}$  and  $\sigma_i$ . Moreover, the arcs added to  $P_{i-1}$  and  $P_i$  at  $x$  to obtain  $\sigma'_{i-1}$  and  $\sigma'_i$  are inside  $C$ . If  $x$  is a rainbow in  $C$ , then  $P_i$  and  $P_{i-1}$  are not extended at  $x$ , and  $x$  acts as one of the degree-2 vertices in Figure 2. The rest of the points in  $\delta(C)$  are crossings in  $\bigcup_{i=0}^m \sigma'_i$  facing the interior of  $C$ . Since  $C$  has at most two rainbows,  $\bigcup_{i=0}^m \sigma'_i$  is one of the drawings depicted in Figure 2. ◀

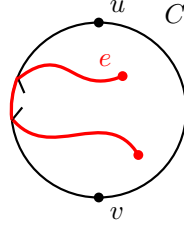
Theorem 2 can also be applied to show that a drawing of  $K_n$  is pseudolinear if and only if does not contain the  $B$ -configuration (Theorem 2.5.1 in [3]). We sketch the proof of a specific case of this theorem in the next two paragraphs and comment on the general case afterwards.

Suppose that  $G(\Sigma)$  is the planarization of a non-pseudolinear drawing  $D$  of  $K_n$  for which we would like to show that  $D$  contains a  $B$ -configuration. Consider an obstruction  $C$  of  $G(\Sigma)$  minimizing  $|\delta(C)|$ , where  $\delta(C)$  are vertices of  $C$  incident with edges in  $C$  having different colours. For illustrative purposes, let us assume that  $C$  contains two vertices from  $V(K_n)$ . Since  $C$  is an obstruction,  $u$  and  $v$  are the only rainbows of  $C$ .

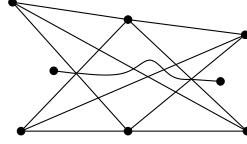
An edge  $e$  of  $K_n$  is *involved* in  $C$  if  $C$  contains a subarc of  $D[e]$  (see Figure 13). By using Lemma 4 is not hard to show that every edge involved in  $C$  is drawn inside  $C$ . Consider all the vertices incident with an edge involved in  $C$  and let  $D'$  be the drawing of the complete graph induced by these vertices. Then,  $D'$  has at most two vertices in its outer boundary, namely  $u$  and  $v$ . Thus, the outer boundary of  $D'$  is incident with at least one crossing. The  $K_4$  containing this crossing is drawn as in Figure 1 with its crossing incident with the outer face. This  $K_4$  contains a  $B$ -configuration.

The proof for the general case, where  $C$  does not necessarily contains two vertices of  $K_n$ , is considered in full detail in [3], and uses the complete subgraph induced by the edges involved in  $C$  combined with the fact that  $|\delta(C)|$  is minimal.

A drawing is *stretchable* if it is homeomorphic to a rectilinear drawing. There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 14. Nevertheless, the following is an immediate consequence of Thomassen's main result in [19].



■ **Figure 13** An edge  $e$  involved in the obstruction  $C$ .



■ **Figure 14** Non-Pappus configuration.

630 ► **Corollary 16.** *A 1-planar drawing of a graph is stretchable if and only if it is pseudolinear.*

631 **Proof.** If a drawing  $D$  is stretchable then clearly it is pseudolinear. To show the converse,  
 632 suppose that  $D$  is pseudolinear. Then  $D$  does not contain any obstruction, and in particular,  
 633 neither of the  $B$ - and  $W$ -configurations in Figure 2 occurs in  $D$ . This condition was shown  
 634 in [19] to be equivalent to being stretchable. ◀

635 One can construct more general examples of pseudolinear drawings that are not stretchable  
 636 by considering non-stretchable arrangements of pseudolines. However, such examples seem to  
 637 inevitably have some edge with multiple crossings. This leads to a natural question.

638 ▷ **Question 17.** Is it true that if  $D$  is a pseudolinear drawing in which every edge is crossed  
 639 at most twice, then  $D$  is stretchable?

640 We believe that there are other instances where pseudolinearity characterizes stretchability  
 641 of drawings. A drawing is *near planar* if the removal of one edge produces a planar graph.  
 642 One instance, is the following result by Eades et al. that can be translated to the language  
 643 of pseudolines:

644 ► **Theorem 18.** [9] *A drawing of a near-planar graph is stretchable if and only if the drawing*  
 645 *induced by the crossed edges is pseudolinear.*

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